Exercise 67

(a) Use a graph to estimate the absolute maximum and minimum values of the function to two decimal places.

(b) Use calculus to find the exact maximum and minimum values.

$$f(x) = x\sqrt{x - x^2}$$

Solution

Since no domain is specified, the function's entire domain is used.

$$x - x^2 \geq 0$$

$$x(1 - x) \geq 0$$

$$0 \leq x \leq 1$$

Take the derivative of the function.

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left(x\sqrt{x - x^2}\right) \\
&= \left[\frac{d}{dx}(x)\right]\sqrt{x - x^2} + x\left[\frac{d}{dx}(\sqrt{x - x^2})\right] \\
&= (1)\sqrt{x - x^2} + x\left[\frac{1}{2}(x - x^2)^{-1/2} \cdot \frac{d}{dx}(x - x^2)\right] \\
&= \sqrt{x - x^2} + x\left[\frac{1}{2}(x - x^2)^{-1/2} \cdot (1 - 2x)\right] \\
&= \sqrt{x - x^2} + \frac{x(1 - 2x)}{2\sqrt{x - x^2}} \\
&= \frac{2(x - x^2) + x(1 - 2x)}{2\sqrt{x - x^2}} \\
&= \frac{3x - 4x^2}{2\sqrt{x - x^2}}
\end{aligned}$$

Set what's in the numerator equal to zero, and set what's in the denominator equal to zero. Solve each equation for x.

$$\begin{aligned}
3x - 4x^2 &= 0 & \qquad 2\sqrt{x - x^2} &= 0 \\
x(3 - 4x) &= 0 & x - x^2 &= 0 \\
x = 0 \quad \text{or} \quad 3 - 4x &= 0 & x(1 - x) &= 0 \\
x = 0 \quad \text{or} \quad x &= \frac{3}{4} & x = 0 \quad \text{or} \quad x &= 1
\end{aligned}$$

$x = 0$ and $x = 3/4$ and $x = 1$ are within $0 \le x \le 1$, so evaluate f at these values.

$$f(0) = (0)\sqrt{0 - 0^2} = 0 \qquad \text{(absolute minimum)}$$

$$f\left(\frac{3}{4}\right) = \left(\frac{3}{4}\right)\sqrt{\left(\frac{3}{4}\right) - \left(\frac{3}{4}\right)^2} = \frac{3\sqrt{3}}{16} \approx 0.32476 \qquad \text{(absolute maximum)}$$

$$f(1) = (1)\sqrt{1 - 1^2} = 0 \qquad \text{(absolute minimum)}$$

The smallest and largest of these numbers are the absolute minimum and maximum, respectively, over the interval $0 \le x \le 1$. The graph below illustrates these results.